Direct and indirect factor income shares at the sectoral level: An alternative method using input-output analysis*

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Abstract

The purpose of the paper is to suggest a method to decompose gross output or total input in an alternative way at aggregate and sectoral levels using elementary input-output analysis. Total input consists of domestic and imported intermediate inputs, and value-added. Domestic intermediate inputs are produced by domestic firms, and thus can, again, be decomposed into the same three components following the input structures of the economy or the industries. The same decomposition can be repeatedly applied, and finally domestic intermediate inputs can be decomposed into 'indirect' value-added and imported intermediate inputs. In conclusion, we can decompose gross output into four components, direct and indirect value-added and imports, or equivalently, direct and indirect, domestic and foreign value-added.
The results of the paper can be compared among countries, among years, among industries, and among groups of industries, and the results can be used to study the characteristics of the industries, the environment of domestic and global markets, and their changes. We applied the decomposition method proposed in the paper to Korea's input-output tables for 1970-2019 at aggregate and industry levels.

Keywords: Direct and indirect factor income, sectoral factor income, input-output analysis, domestic outsourcing, global offshoring.

JEL Classification: D57, D33

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1 Motivation and Literature

In this paper we will propose an alternative method to estimate the factor income shares at the sectoral level.[1] Specifically, we will decompose the gross output of individual industries into the compensation for various primary production factors, both domestic and foreign.

To understand the concept, we need a brief review of the system of national accounts such as national income statistics and input output tables. Gross output is the total amount of the sales of all goods and services produced and sold in a period, usually a year, evaluated at market price. Roughly speaking, it is the total sales of all products by all firms in a country in a year.

Firms use two types of production factors: intermediate inputs and primary production factors. Intermediate input (II), also referred to as intermediate consumption, is the amount of the compensation for the goods and services consumed in the firms' production process. They are the firms' expenditure on raw materials. IIs are classified into domestic and imported ones.

Market value of gross output is bigger than that of II, and the difference is the increase in the value generated by firms' production process and is called value-added (VA). The second category of production factors is the service of labor and capital, basically provided by households. They are called 'primary' production factors and are different from IIs in that they are not produced by firms. VA is paid to the providers of the primary production factors. Figure 1 depicts the decomposition of gross output into domestic and imported IIs and VA.

[1] The terms 'industry' and 'sector' will be used interchangeably in this paper. In the later part of this paper, however, the term 'sector' will be used as a wider concept, consisting of industries. For example, we will refer to the group of manufacturing 'industries' as the 'manufacturing sector.' Also, in IO analysis, the distinction between 'products' and 'industries' can become blurred in certain situations, and we will use these terms flexibly.

Figure 1. Composition of Gross Output

<table>
<tr><td rowspan="3">Gross Output</td><td rowspan="2">Intermediate Input</td><td>Domestic Intermediate Input</td></tr>
<tr><td>Imported Intermediate Input</td></tr>
<tr><td>Value-Added</td><td></td></tr>
</table>

According to the system of national accounts, total VA is divided into several components. In Korea's input-output (IO) tables, for example, total VA consists of five components; wages and salaries, operating surplus, consumption of fixed capital, production tax and subsidy. Wages and salaries are the payment to employees, i.e., paid workers, and consumption of fixed capital is the compensation for the decrease in the book value of capital due to depreciation, and paid to the providers of capital. Operating surplus is computed as the residual after other four components are paid and include the payment for the service of capital such as interest payment and dividend, payment to nonpaid workers such as self-employed and family workers, and so forth. Components of VA are called the factor incomes. While aggregate factor incomes can be obtained from the national income accounts, those at the sectoral level are available from IO tables.

Imported IIs have a clear distinction from domestic IIs. Imported IIs are classified as 'exports' in the exporting countries' national accounts. Exports are a component of the final demand of a country, and thus is a part of VA. For this reason, imports can be referred to as 'foreign' VA.

In consequence, domestic IIs are the only component of gross output that are not a part of domestic or foreign VA. Now, using the input structures of individual domestic industries, we can convert the domestic IIs into the payments to domestic IIs, imported IIs and VA. This is possible because domestic IIs are produced by domestic firms and their input structures are given in IO tables.

This process can be repeated infinitely, and finally total gross output can be expressed as the sum of VA and imported IIs, or equivalently, domestic VA and foreign VA. The initial amounts of VA and imported IIs can be referred to as 'direct' factor incomes while those computed in the subsequent steps as 'indirect' factor incomes.

The shares of these four components – direct and indirect, and domestic and foreign factor incomes – show significant variations among industries. They show variation not only in the shares of direct factor incomes, but also in the shares of indirect factor incomes due to various inter-industry relationships. These shares are the result of firms' profit maximization and reflect industries' characteristics and constraints they face. Therefore, the results of the paper can be useful in studying the various characteristics of individual industries, the domestic and international environment they face, the industries' responses to the environmental changes, and so on.

It is possible to further decompose imported IIs into factor incomes if we have the IO tables of all our trade partner countries in a common layout and common currency unit. In that case, we can decompose total gross output into direct and indirect, domestic and foreign factor incomes. However, the significance and the usefulness of these results as well as the data availability appear questionable. It is because the shares of direct and indirect imported IIs might provide more useful information about the environments and trends of the international markets.

IO tables are the almost only source of statistics for the purpose of the paper, and the method suggested in the paper requires relatively simple IO analysis. Basically, it is similar with the method for computing the impact of final demand shocks on the supply-side variables. We used the IO tables of Korea for 1970-2019 in this paper for empirical results.

Most previous literature on factor incomes primarily focused on the issues related with income distribution. Specifically, they have focused on the relative sizes of labor and capital income shares and their time-series trend. For that reason, much effort was exerted on 'estimating' the compensation for unpaid workers, which is not observed thus needs to be estimated. This is true for the studies at both aggregate and sectoral levels.

While the studies in factor income shares at aggregate level are countless, almost all of them utilize the statistics available in the national income and product accounts, a part of the System of National Accounts (SNA). The three major issues in this field are; (i) how to estimate the labor income of unpaid workers (the numerator), specifically, how to decompose operating surplus into the labor income of unpaid workers and capital income, (ii) whether to include labor income from abroad in the total labor income (the numerator and the denominator), and (iii) the coverage of total factor income (the denominator), that is, whether to include or exclude various income components in total factor income, such as net factor income from abroad, consumption of fixed capital, net indirect taxes, operating surplus of private unincorporated enterprises (OSPUE), etc.

The work by Gollin (2002) has been frequently cited regarding the first issue. He emphasized the importance of OSPUE in estimating the labor income share and suggested three options to classify OSPUE into labor and capital incomes under various assumptions. The options by Gollin were utilized in estimating the labor income share in many countries, for example, by Kim (2013), Joo and Jeon (2014), Lee (2014), Im (2020), etc. in Korea. Bank of Korea publishes Korea's 'official' labor income share statistics. It regards wages and salaries and net labor income from abroad as the country's total labor income, and the labor income share by Bank of Korea is lower than all options suggested by Gollin.

Krueger (1999) suggested four options which depend on the ideas of Gollin (2002) and Johnson (1954) and applied his formulas to the US economy. OECD and ILO suggested several variations of the above-mentioned formulas and applied them to their member countries, which can be found in ILO and OECD (2015), ILO (2019), etc.

The research on factor income shares at the sectoral level is highly limited. The researchers in this field face two big obstacles. The first is the lack of statistics because many components in the formulas are available only at aggregate levels. Net labor income from abroad is a typical example. The second obstacle is the big variations in the shares of unpaid workers and average wage levels of paid workers. Many formulas apply the average wage levels of paid workers to unpaid workers in extracting the labor income of unpaid workers from operating surplus, but sometimes these formulas result in the labor income of unpaid workers being bigger than operating surplus which makes it impossible to apply the formula.

Some of recent works in this field are briefly introduced here.[2] Bentolila and Saint-Paul (2003) estimated the labor income shares of twelve countries at the sectoral level for 1972-1993 using country-industry panel data. Zuleta and Young (2007) and Young (2010) estimated the labor income shares of the United States at the sectoral level for 1958-1996 using industry data. Valentiny and Herrendorf (2008) estimated the labor income shares of the United States at the sectoral level using IO tables. Im (2020) estimated the

[2] This paragraph is based on Kim (2021).

labor income shares of Korea at the sectoral level for 2015–2018 using IO tables and household survey data. He applied various formulas in estimating the labor income of unpaid workers. Kim (2021) estimated the labor income shares of Korea at the sectoral level for a long period of 1975–2018 using IO tables. He rearranged all IO tables for the period according to a common layout and common industry classification so that common formulas can be applied. He estimated the labor income shares by various formulas and compared the results with previous research.

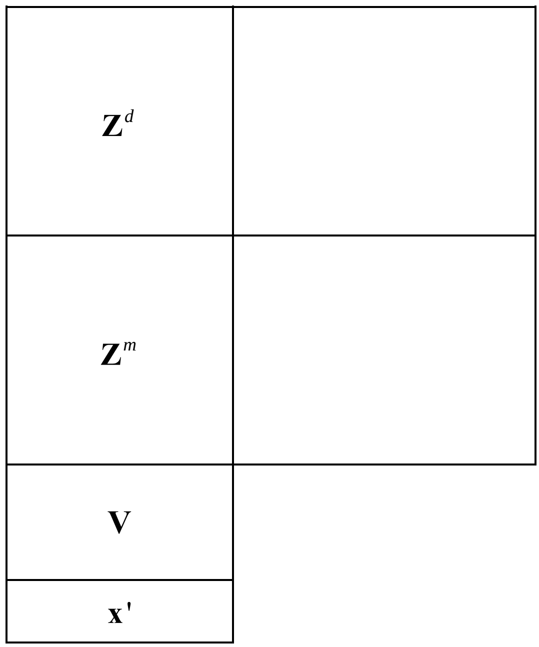
While all the above works estimated the factor income shares at the aggregate or sectoral level, they were mainly interested in the share of labor income in national income. Unlike previous works, the purpose of this paper is to decompose total gross output into direct and indirect factor incomes. For that reason, we do not attempt to extract the labor income of unpaid workers to estimate the total labor income share. Instead, we will decompose domestic II into indirect domestic and foreign factor incomes.

The paper is organized as follows. Section II is the main part of the paper, and it explains the IO analysis for decomposing gross output into direct and indirect, domestic and foreign VA. Section III explains the data and provides the empirical results, and Section IV concludes the paper.

2 Decomposition of Gross Output into Domestic and Foreign Value-Added

Let n be the number of products/industries and assume that we have arranged the IO tables as in Figure 2. In Figure 2, $\boldsymbol{Z}^d = [z_{ij}^d]$ and $\boldsymbol{Z}^m = [z_{ij}^m]$ are the $n \times n$ endogenous sectors of the domestic and imported tables, respectively, that is, the inter-industry transaction matrices of domestic and imported goods and services among industries to be used as IIs. Specifically, z_{ij}^d and z_{ij}^m are the amounts of the domestic and imported ith products used as raw material in the jth industry. We will use the indices $i = 1,2, \ldots, n$ and $j = 1,2, \ldots, n$ for products (rows) and industries (columns), respectively.

Figure 2. Layout of Input-Output Tables



Assume that total VA consists of p components. For example, total VA in Korea's IO tables consists of four components – wages and salaries, operating surplus, consumption of fixed capital and net production tax – and thus $p = 4$. Total VA in the IO tables in the 2018 Edition of OECD's Input-Output Database consists of three components – labor costs: compensation of employees, other taxes less subsidies on production, and gross operating surplus and mixed income – and thus $p = 3$. Then $\boldsymbol{V} = [v_{kj}]$ is the $p \times n$ VA matrix. We will use the index $k = 1,2, \ldots, p$ for the components of VA. Finally, the $n \times 1$ vector of gross outputs is denoted by $\boldsymbol{x}$, and the $1 \times n$ vector of total inputs by $\boldsymbol{x}'$. The total input row vector and the transpose of gross output column vector must be identical in IO tables. Final demand sector of IO tables is not used in the paper.

Let $\boldsymbol{o}$ and $\boldsymbol{o}_p$ be the $n \times 1$ and $p \times 1$ vectors of 1s, respectively. Considering that total input consists of domestic and imported IIs and VA, the vertical sums of $\boldsymbol{Z}^d$, $\boldsymbol{Z}^m$ and $\boldsymbol{V}$ must add up to the total input row, i.e.,

$$\mathbf{o}'\mathbf{Z}^d + \mathbf{o}'\mathbf{Z}^m + \mathbf{o}'_p\mathbf{V} = \mathbf{x}'. \tag{1}$$

Let $\boldsymbol{z}_j^d$, $\boldsymbol{z}_j^m$ and $\boldsymbol{v}_j$ be the jth columns of $\boldsymbol{Z}^d$, $\boldsymbol{Z}^m$ and $\boldsymbol{V}$, respectively, and let x_j be the jth element of $\boldsymbol{x}'$. Then $\mathbf{z}_j^d$, $\mathbf{z}_j^m$ and $\boldsymbol{v}_j$ are the domestic and imported inputs and VA vectors in the jth industry while x_j is the total input in the jth industry, and we have

$$\mathbf{o}'\mathbf{z}_j^d + \mathbf{o}'\mathbf{z}_j^m + \mathbf{o}'_p\mathbf{v}_j = x_j, j = 1,2, \dots, n. \tag{2}$$

Equation (2) is the jth element of (1), and the left-hand side is the decomposition of the jth industry's total input/gross output.

Define the input coefficient matrices as follows:

$$\mathbf{A}^d = [a_{ij}^d] = [z_{ij}^d / x_j] = \mathbf{Z}^d \,./\, \mathbf{x}',$$

$$\mathbf{A}^m = [a_{ij}^m] = [z_{ij}^m / x_j] = \mathbf{Z}^m ./\, \mathbf{x}',$$

$$\mathbf{A}^v = [a_{kj}^v] = [v_{kj} / x_j] = \mathbf{V} \;./\, \mathbf{x}''$$

where './' is the element-by-element (EBE) division operator.[3] Divide both sides of (1) by $\boldsymbol{x}'$, and we get

$$\mathbf{o}'\mathbf{A}^d + \mathbf{o}'\mathbf{A}^m + \mathbf{o}'_p\mathbf{A}^v = \mathbf{o}', \tag{3}$$

and the jth element of (3) can be written as

$$\mathbf{o}'\mathbf{a}_j^d + \mathbf{o}'\mathbf{a}_j^m + \mathbf{o}'_p\mathbf{a}_j^v = 1, j = 1,2, \dots, n, \tag{4}$$

where $\boldsymbol{a}_j^d$, $\boldsymbol{a}_j^m$ and $\boldsymbol{a}_j^v$ are the jth columns of $\boldsymbol{A}^d$, $\boldsymbol{A}^m$ and $\boldsymbol{A}^v$, respectively. (4) can also be derived by dividing both sides of (2) by x_j, $j = 1,2, \dots, n$. Equations (1) and (2) describe the input structures of all industries and of the jth industry in 'amounts,' respectively, while (3) and (4) describe them in 'shares.' Equations (3) and (4) are usually referred to as the input

[3] Let $\boldsymbol{a} = [a_i]$ and $\boldsymbol{b} = [b_i]$ be vectors with same size. Then the EBE division of $\boldsymbol{a}$ by $\boldsymbol{b}$ is defined as $\boldsymbol{a}./\boldsymbol{b} = [a_i/b_i]$ where ./ is the EBE division operator.

structure of each industry, which tells us the amounts of domestic and imported IIs ($n + n$ terms) and VA terms (p terms) necessary for producing one unit of individual products.

The terms on the left-hand side of (2) are the amounts of domestic IIs, imported IIs and VA required to produce the jth product by x_j. Of these requirements, imported IIs ($\boldsymbol{o}'\boldsymbol{z}_j^m$) is supplied by the foreign sector, and VA ($\boldsymbol{o}'_p\boldsymbol{v}_j$) is supplied by households. Domestic IIs ($\boldsymbol{o}'\boldsymbol{z}_j^d$), on the other hand, must be produced by domestic firms.[4] Note that $\boldsymbol{o}'\boldsymbol{z}_j^d$ can be rewritten as

$$\mathbf{o}'\mathbf{z}_j^d = z_{1j}^d + z_{2j}^d + \cdots + z_{nj}^d = \sum_{i=1}^{n} z_{ij}^d . \tag{5}$$

In other words, we need to produce domestic products $i = 1,2,\ldots,n$ by the amounts $z_{1j}^d, z_{2j}^d, \ldots, z_{nj}^d$ to be used as IIs in order to produce the jth product by x_j.

These domestic IIs must be produced by domestic firms, and again we need domestic and imported IIs and VA according to the input structure (4). Thus, the amounts of the production factors required in producing z_{ij}^d are

$$z_{ij}^d = z_{ij}^d \left(\sum_{l=1}^{n} a_{li}^d + \sum_{l=1}^{n} a_{li}^m + \sum_{k=1}^{p} a_{ki}^v \right). \tag{6}$$

Substituting (6) into (5), we get the following result.

$$\mathbf{o}'\mathbf{z}_j^d = \sum_{i=1}^{n} z_{ij}^d \left(\sum_{l=1}^{n} a_{li}^d + \sum_{l=1}^{n} a_{li}^m + \sum_{k=1}^{p} a_{ki}^v \right) = \mathbf{o}'\mathbf{A}^d\mathbf{z}_j^d + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d + \mathbf{o}'_p\mathbf{A}^v\mathbf{z}_j^d . \tag{7}$$

See Appendix for the proof of (7). Note that we can implement (7) for any vector of domestic products. We can modify the equation (2) utilizing (7) to get

$$(2)' \quad x_j = \mathbf{o}'\mathbf{z}_j^d + \mathbf{o}'\mathbf{z}_j^m + \mathbf{o}_p{}'\mathbf{v}_j$$

$$= (\mathbf{o}'\mathbf{A}^d\mathbf{z}_j^d + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d + \mathbf{o}'_p\mathbf{A}^v\mathbf{z}_j^d) + \mathbf{o}'\mathbf{z}_j^m + \mathbf{o}_p{}'\mathbf{v}_j \qquad \text{by (7)}$$

[4] In IO analysis, it is assumed that the supply side is infinitely elastic, and every demand is fulfilled instantaneously.

$$= \mathbf{o}'\mathbf{A}^d\mathbf{z}_j^d + (\mathbf{o}'\mathbf{z}_j^m + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d) + (\mathbf{o}'_p\mathbf{v}_j + \mathbf{o}'_p\mathbf{A}^v\mathbf{z}_j^d),. \qquad j = 1,2,\dots,n$$

This is the result of applying the previous decomposition twice. Direct impact of x_j on domestic IIs is $\mathbf{z}_j^d$, the first terms in the parentheses are the direct impact of x_j on imported IIs and VA, respectively, and the second terms are the second-round indirect impact of $\mathbf{z}_j^d$. The first term, $\boldsymbol{o}'\boldsymbol{A}^d\boldsymbol{z}_j^d$, is the second-round indirect impact on domestic IIs. We can apply the previous decomposition infinitely many times, and (2)' becomes

$$
\begin{aligned}
(2)'' \quad x_j &= \mathbf{o}'\mathbf{z}_j^d + \mathbf{o}'\mathbf{z}_j^m + \mathbf{o}_p'\mathbf{v}_j \\
&= \mathbf{o}'\mathbf{A}^d\mathbf{z}_j^d + (\mathbf{o}'\mathbf{z}_j^m + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d) + (\mathbf{o}_p'\mathbf{v}_j + \mathbf{o}_p'\mathbf{A}^v\mathbf{z}_j^d) \\
&= \mathbf{o}'\mathbf{A}^{d2}\mathbf{z}_j^d + (\mathbf{o}'\mathbf{z}_j^m + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d + \mathbf{o}'\mathbf{A}^m\mathbf{A}^d\mathbf{z}_j^d) + (\mathbf{o}_p'\mathbf{v}_j + \mathbf{o}_p'\mathbf{A}^v\mathbf{z}_j^d + \mathbf{o}_p'\mathbf{A}^v\mathbf{A}^d\mathbf{z}_j^d) \\
&= \mathbf{o}'\mathbf{A}^{d3}\mathbf{z}_j^d + (\mathbf{o}'\mathbf{z}_j^m + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d + \mathbf{o}'\mathbf{A}^m\mathbf{A}^d\mathbf{z}_j^d + \mathbf{o}'\mathbf{A}^m\mathbf{A}^{d2}\mathbf{z}_j^d) \\
&\qquad + (\mathbf{o}_p'\mathbf{v}_j + \mathbf{o}_p'\mathbf{A}^v\mathbf{z}_j^d + \mathbf{o}_p'\mathbf{A}^v\mathbf{A}^d\mathbf{z}_j^d + \mathbf{o}_p'\mathbf{A}^v\mathbf{A}^{d2}\mathbf{z}_j^d) \\
&= \mathbf{o}'\mathbf{z}_j^m + \mathbf{o}'\mathbf{A}^m(\mathbf{I} + \mathbf{A}^d + \mathbf{A}^{d2} + \cdots)\mathbf{z}_j^d + (\mathbf{o}_p'\mathbf{v}_j + \mathbf{o}_p'\mathbf{A}^v(\mathbf{I} + \mathbf{A}^d + \mathbf{A}^{d2} + \cdots)\mathbf{z}_j^d \\
&= \underbrace{\mathbf{o}'\mathbf{z}_j^m}_{\substack{\text{direct}\\ \text{import}}} + \underbrace{\mathbf{o}'\mathbf{A}^m(\mathbf{I} - \mathbf{A}^d)^{-1}\mathbf{z}_j^d}_{\substack{\text{indirect}\\ \text{import}}} + \underbrace{\mathbf{o}_p'\mathbf{v}_j}_{\substack{\text{direct}\\ \text{value-added}}} + \underbrace{\mathbf{o}_p'\mathbf{A}^v(\mathbf{I} - \mathbf{A}^d)^{-1}\mathbf{z}_j^d}_{\substack{\text{indirect}\\ \text{value-added}}}, \quad j = 1, 2, \dots, n.
\end{aligned}
$$

We can concatenate (2)'' horizontally for $j = 1,2,\dots,n$, and finally we obtain

$$
\begin{aligned}
\mathbf{x}' &= \mathbf{o}'\mathbf{Z}^m + \mathbf{o}'\mathbf{A}^m(\mathbf{I} - \mathbf{A}^d)^{-1}\mathbf{Z}^d + \mathbf{o}'_p\mathbf{V} + \mathbf{o}'_p\mathbf{A}^v(\mathbf{I} - \mathbf{A}^d)^{-1}\mathbf{Z}^d \\
&= \mathbf{o}'\mathbf{Z}^m + \mathbf{o}'\mathbf{R}^m\mathbf{Z}^d + \mathbf{o}'_p\mathbf{V} + \mathbf{o}'_p\mathbf{R}^v\mathbf{Z}^d, \qquad (8)
\end{aligned}
$$

$\mathbf{R}^v = \mathbf{A}^v(\mathbf{I} - \mathbf{A}^d)^{-1}$ and $\mathbf{R}^m = \mathbf{A}^m(\mathbf{I} - \mathbf{A}^d)^{-1}$.

This is the result of the decomposition of gross output into direct and indirect VA (domestic VA) and imports (foreign VA). Upon dividing both sides of (8) by $\boldsymbol{x}'$, we get

$$\mathbf{o}' = \mathbf{o}'\mathbf{A}^m + \mathbf{o}'\mathbf{R}^m\mathbf{A}^d + \mathbf{o}'_p\mathbf{A}^v + \mathbf{o}'_p\mathbf{R}^v\mathbf{A}^d. \qquad (9)$$

Equations (8) and (9) express the decompositions in amounts and in shares, respectively.[5] We can tabulate the decompositions as in Table 1.

[5] The proofs of the equalities in (8) and (9) are given in the Appendix.

Table 1. Decomposition of Gross Output into Direct and Indirect Factor Incomes

			Direct Impact	Indirect Impact	Total
Decomposition in amounts	Value-Added	(Domestic VA)	$\mathbf{o}_p'\mathbf{V}$	$\mathbf{o}_p'\mathbf{R}^v\mathbf{Z}^d$	$\mathbf{x}'$
	Imports	(Foreign VA)	$\mathbf{o}'\mathbf{Z}^m$	$\mathbf{o}'\mathbf{R}^m\mathbf{Z}^d$	
Decomposition in shares	Value-Added	(Domestic VA)	$\mathbf{o}_p'\mathbf{A}^v$	$\mathbf{o}_p'\mathbf{R}^v\mathbf{A}^d$	$\mathbf{o}'$
	Imports	(Foreign VA)	$\mathbf{o}'\mathbf{A}^m$	$\mathbf{o}'\mathbf{R}^m\mathbf{A}^d$	

$\mathbf{R}^v = \mathbf{A}^v(\mathbf{I}-\mathbf{A}^d)^{-1}$, $\mathbf{R}^m = \mathbf{A}^m(\mathbf{I}-\mathbf{A}^d)^{-1}$.

Note that $\boldsymbol{R}^v = \boldsymbol{A}^v(\boldsymbol{I}-\boldsymbol{A}^d)^{-1}$ and $\boldsymbol{R}^m = \boldsymbol{A}^m(\boldsymbol{I}-\boldsymbol{A}^d)^{-1}$ are the VA and imports multiplier matrices of the demand for domestic products. Thus the indirect impacts in the above table are the amounts of VA and imports generated by the intermediate demand for domestic products. In conclusion, gross output or total input consists of VA (direct domestic VA), imported IIs (direct foreign VA), and domestic IIs, and we can decompose domestic IIs into indirect VA (indirect domestic VA) and imported IIs (indirect foreign VA). These results are highly intuitive, and we could have derived the above decomposition using simple IO analysis.

One of the remarkable advantages of the method in this paper is that the decomposition results can be compared among countries, among years, among industries, and among group of industries. Comparison among countries illustrates the degree of domestic and international outsourcing, more specifically, the extents to which countries or industries utilize domestic and global production networks. Comparison among years, on the other hand, explains the changes in the characteristics of the economies and the environments.

3 Data and Empirical Results

In this paper, we used the IO tables of Korea for 1970-2019. Korea has produced IO tables 35 times since 1960. The IO tables in 1960s were Korea's first attempts and are not usable or reliable enough for empirical studies. The IO tables since 1970s were produced with rigor and are available in electronic forms and were used in this paper. All tables were rearranged according to the same 26-industry classification and same layout so that common formulas can be applied. Tables of some years, unfortunately, could not be arranged for several reasons, and 22 years' tables were used in this paper.[6] The industry classification of this paper is given in Table 1. The tables evaluated at producers' price were used in this paper.

Table 2. Industry Classification

Number	Name
1	Agricultural, forest, and fishery goods
2	Mined and quarried goods
3	Food, beverages and tobacco products
4	Textile and leather products
5	Wood and paper products
6	Coal and petroleum products
7	Chemical products
8	Non-metallic mineral products
9	Basic metal products
10	Fabricated metal products
11	Machinery and equipment
12	Electrical and electronic equipment and components
13	Precision instruments
14	Motor vehicles
15	Other transport equipment
16	Other manufactured products
17	Electricity, gas, and water supply
18	Construction
19	Wholesale and retail
20	Food services and accommodation
21	Transportation and warehousing
22	Communications and broadcasting
23	Finance, insurance, real estate services and business services
24	Public administration and defense
25	Education, research and healthcare services
26	Other services

[6] The years of the tables used in this paper are 1970, 1973, 1975, 1980, 1983, 1985, 1990, 1995, 1998, 2000, 2003, 2005, 2006, 2007, 2008, 2009, 2010, 2015, 2016, 2017, 2018, 2019.

The formulas in the equations (8) and (9) were used to decompose total input/gross output into four factor incomes: direct and indirect, domestic and foreign VA. Foreign VA is divided into $n = 26$ components, but only the total is reported here. Domestic VA is divided into $p = 4$ components. The decomposition formulas were applied at the aggregate ($n = 1$) and industry ($n = 26$) levels. Also, the results with 3-industry classification ($n = 3$) are reported here: that is, (i) agriculture, forestry, and fishery, (ii) mining and manufacturing, and (iii) service sectors.

Decomposition of total input into direct and indirect factor incomes for Korea in 2019 is given in Table 3. In 2019, the shares of direct VA and imports were 43.5% and 12.3%, respectively. This implies that the remaining 44.2% was spent on domestic IIs, which was decomposed into indirect VA (34.4%) and indirect imports (9.7%).

Table 3. Direct and Indirect Factor Income Shares of Korea in 2019

Industry	Factor Income Share					I/(D+I) Ratio	
	Value-Added		Imports		Total Input	Value-Added	Imports
	Direct	Indirect	Direct	Indirect			
Aggregate	43.5	34.4	12.3	9.7	100.0	44.2	44.2
Agriculture etc.	51.0	33.6	3.2	12.2	100.0	39.7	79.1
Manufacturing	29.1	35.3	21.5	14.1	100.0	54.8	39.5
Service	53.9	31.8	5.8	8.6	100.0	37.1	59.8
1	51.0	34.0	3.2	11.7	100.0	40.0	78.5
2	47.2	38.4	0.6	13.9	100.0	44.8	96.1
3	25.6	50.1	11.7	12.5	100.0	66.2	51.6
4	20.3	33.6	32.2	13.8	100.0	62.3	30.1
5	28.9	40.7	15.0	15.4	100.0	58.4	50.5
6	25.1	10.3	59.9	4.6	100.0	29.2	7.2
7	27.3	32.9	22.6	17.2	100.0	54.7	43.2
8	30.6	43.7	8.0	17.7	100.0	58.8	68.9
9	18.7	30.9	32.9	17.5	100.0	62.2	34.7
10	35.6	38.9	6.8	18.7	100.0	52.2	73.3
11	30.9	38.9	13.6	16.6	100.0	55.7	55.1
12	37.1	25.9	24.8	12.2	100.0	41.0	33.1
13	35.2	34.2	16.1	14.5	100.0	49.3	47.5
14	22.4	47.5	9.3	20.8	100.0	67.9	69.1
15	22.4	43.0	17.1	17.4	100.0	65.7	50.4
16	45.3	36.5	4.9	13.3	100.0	44.6	73.0
17	28.7	20.1	38.6	12.7	100.0	41.2	24.8

18	44.2	38.5	3.6	13.7	100.0	46.6	79.0
19	53.3	34.3	3.2	9.2	100.0	39.2	74.5
20	34.2	48.0	5.2	12.6	100.0	58.4	70.7
21	35.6	29.7	18.7	16.0	100.0	45.5	46.0
22	44.3	40.2	7.7	7.8	100.0	47.6	50.5
23	62.3	29.1	2.7	5.9	100.0	31.8	68.6
24	76.8	15.5	3.3	4.4	100.0	16.8	57.5
25	61.0	27.0	3.4	8.6	100.0	30.7	71.8
26	48.3	37.9	3.3	10.5	100.0	44.0	76.2

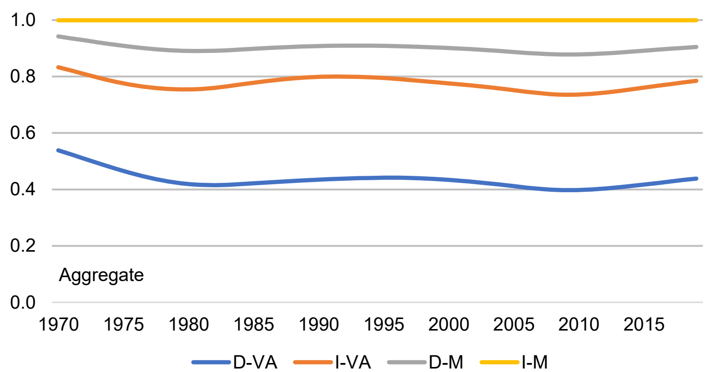
We observe big variations in these shares among various industries. For example, the share of direct VA is small in the manufacturing sector while the shares of direct and indirect imports are relatively big. This phenomenon is strong in major exporting industries such as coal and petroleum products, chemical products, basic metal products, electrical and electronic equipment and components, other transport equipment, etc. Total dependence on foreign VA is relatively low in the service sector. The share of direct and indirect imports in the service sector is 14.4%, much lower than 35.6% in the manufacturing sector. We observe that the industries with high dependence on energy resources also show big shares of total imports: such as petroleum products, basic metal products, electricity, gas, and water supply, transportation and warehousing, and so on.

The shares of indirect income in total income are computed in the last two columns in Table 3. These were computed as the ratio of direct income in total income and were computed for domestic VA and foreign VA separately. This share for VA can be interpreted as a measure of domestic outsourcing while the share for imports can be interpreted as a measure of international offshoring. It is interesting to observe a big difference between manufacturing and service sectors. The results in Table 3 imply that while the manufacturing sector's dependence on foreign VA was relatively higher than the service sector, its dependence on global outsourcing was relatively weaker than its dependence on domestic outsourcing. The pattern of the service sector is the opposite.

As mentioned earlier, the factor income shares and the I/(D+I) ratios can be easily computed for individual VA and import components. They are not reported in the paper, however, due to the limit of the space. This information can be highly valuable when studying the characteristics of specific industries and their environment.

Figure 3 depicts the decomposition of total input into direct and indirect factor incomes for Korea during 1970-2019. Direct VA occupies the biggest share. It was about 54% in 1970, but it declined to below 40% in late 2000s and recovered to about 44% in 2019. The share of indirect VA was about 30% in 1970, and gradually ascended to about 35% in mid-1980s and has stayed at that level since then.

Figure 3. Direct and Indirect Factor Income Shares: Aggregate[7][8]



Note: D=direct, I=indirect, VA=value-added, M=imports.

Observe from Figure 3 that the share of total VA – sum of direct and indirect VA – shows a clear W shape,[9] [10] which narrates the history of Korea's export-led growth strategy since 1970s. Korea began its rapid economic growth in 1970s, but Korea did not have sufficient capacity to produce all IIs necessary for expanding exports and domestic final demand. This led to an increase in imports, mostly of IIs, which explains the first decline of total VA until mid-1980s. 1970s was also the period when Korea began its efforts for import substitution of intermediate goods, which began being effective in mid-1980s. This caused the rise of total VA since 1980s.

The worldwide wave of offshoring and global integration began to become apparent in mid-1990s. Also, Korea implemented country-wide trade liberalization at about the same time, and the overall tariff rates were

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[7] Tables and Figures in this paper were constructed based on the author's computation using Korea's IO tables from Bank of Korea.

[8] Graphs of time-series variables are based on the estimates by Hodrick-Prescott filtering.

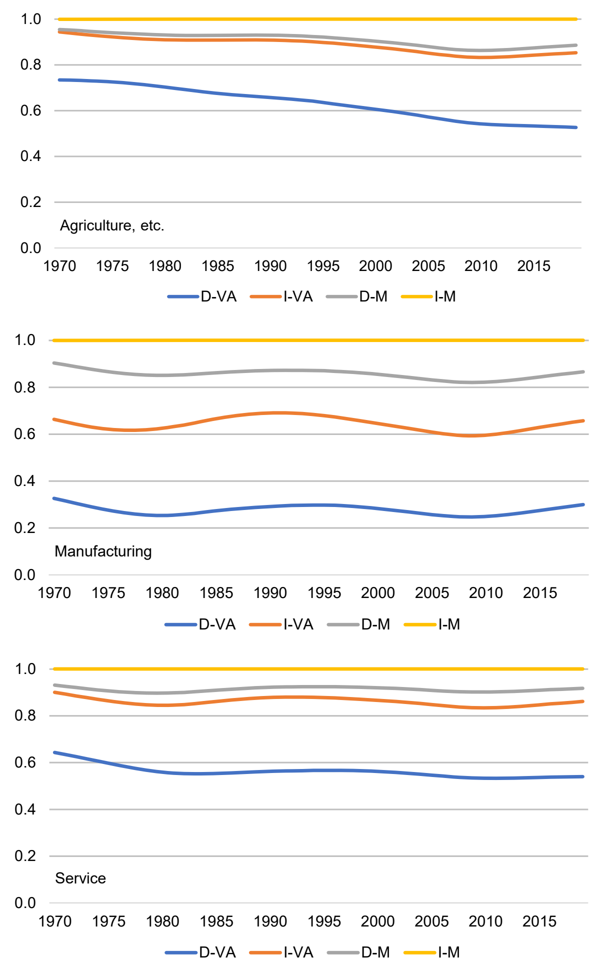
[9] The W shape of the share of total VA implies the M shape of the share of total imports.

[10] This was also mentioned in Kim (2023).

lowered. These caused an increase in the imports of intermediate goods, and the second decrease in the share of total VA since mid-1990s. It is likely that the second upturn of the share of total VA since late 2010s could be due to the change in the global environment and the global trend of onshoring.

Figure 4 depicts the same decomposition at sector level during 1970-2019. It is interesting to observe remarkable differences among the decomposition in the three sectors. In the agriculture, forestry and fishery sector, the share

Figure 4. Direct and Indirect Factor Income Shares: Sector Level

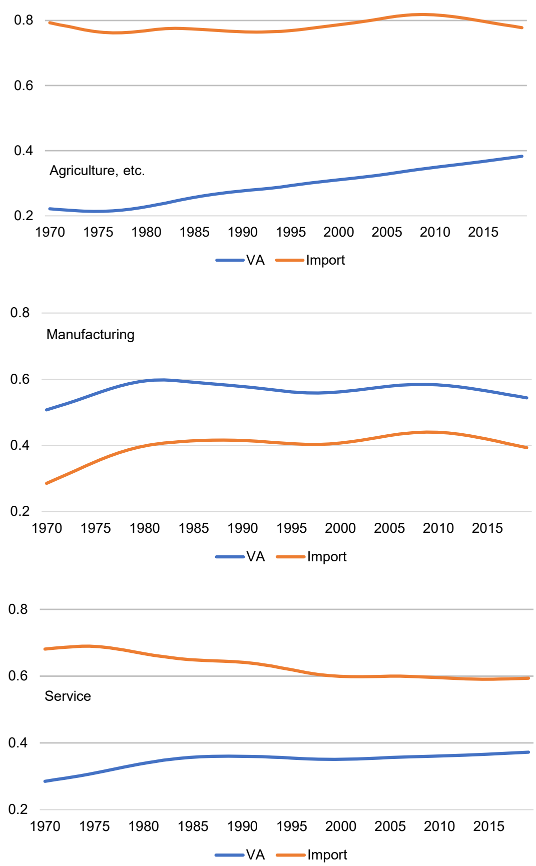


Note: D=direct, I=indirect, VA=value-added, M=imports.

of direct VA has decreased persistently, from about 73% to 51%, while the share of indirect VA has increased during the entire period, from 22% to 33%. This implies that domestic outsourcing has gradually progressed in the agricultural sector. The patterns of the manufacturing and the service sectors are highly close to the aggregate pattern, and we observe clear W and M shapes for total VA and imports. A difference between the manufacturing and the service sectors is that the share of total imports in the service sector is much lower.

The shares of indirect income in total income of the three sectors for 1970-2019 are depicted in Figure 5. Again, these ratios were computed for

Figure 5. Share of Indirect Factor Income ((I/D+I) Ratio): Industry Level



Note: VA=value-added, M=imports.

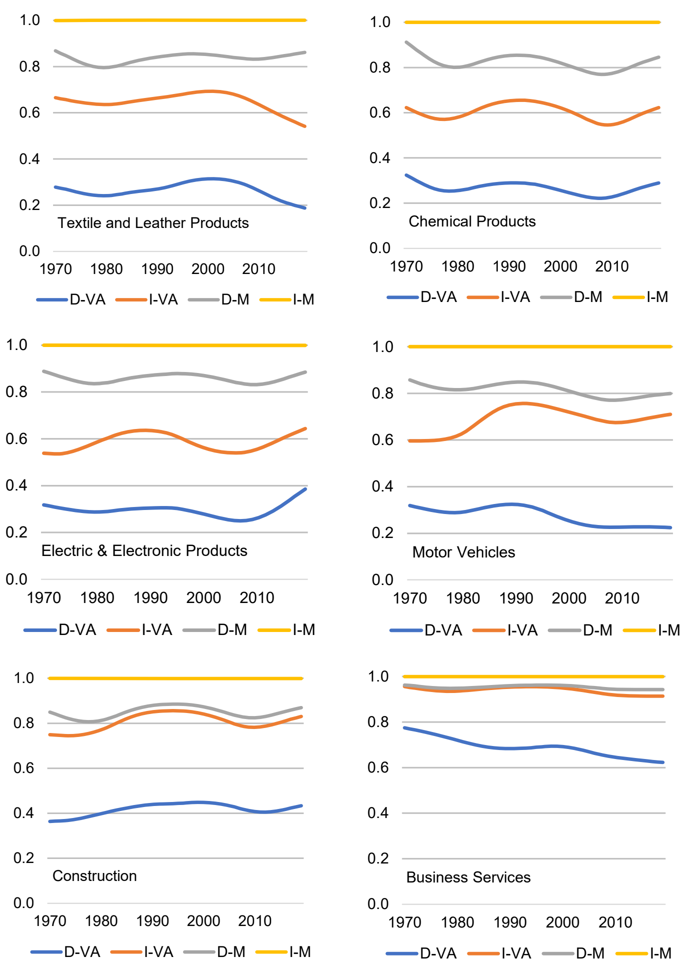
domestic and foreign VAs separately, and they are the indicators of domestic outsourcing and global offshoring, respectively. As in Figure 4, clear differences among the patterns of the three sectors are observed. The degrees of domestic outsourcing (I/(D+I) ratio of domestic VA, blue curves) in the agriculture and the service sectors have been similar in terms of both level and time-series. They have inclined over the entire period from 20-30% up to slightly lower than 40%. On the other hand, the degree of domestic outsourcing in the manufacturing sector shows a clear M shape, which is believed to be related the W shape of the factor income share.

The degree of global offshoring of the agricultural sector has been higher than the other sectors and has been about 80% over the entire period. The degree of global offshoring of the manufacturing sector displays an M shape, which is closely related with the previous explanation.

The decomposition of total input into direct and indirect factor incomes for six selected industries are depicted in Figure 6. Note that a W shape, clear or vague, is observed for the shares of total VA in total input. (i) Textile and leather products industry was selected because it was a leading exporting industry in 1970, and we can see that unlike other industries, the share of total VA has declined since early 2000s and is not recovering, and that the movement of the time-series is lagging behind other industries. It is conjectured that the industry is still heavily depending on the global production network. (ii) Chemical industry is one of Korea's major exporting industries, and the pattern of the decomposition is very similar to that of the entire manufacturing sector in Figure 4. (iii) Electric and electronic products industry is also one of Korea's major exporting industries, and its decomposition pattern is highly similar with that of the manufacturing sector, with two differences: the share of direct VA recovered in recent years, and the share of direct import is bigger than in other industries.

(iv) Motor vehicles industry is also a major exporting industry in Korea, but shows a big difference. The share of domestic VA has persistently declined, almost close to 20% in recent years, while the share of indirect VA is significantly big. It was about 28% in 1970, but reached to almost 50% in 2019. This might signify a remarkable accomplishment in the international competitiveness of parts and components and the resulting high degree of domestic outsourcing. (v) Construction industry is a typical domestic demand-oriented industry, which explains the high shares of direct and indirect VA and a very small share of direct imports. (vi) Business service was selected for its importance as a factor for international competitiveness. It is observed that the share of total imports is very small compared to other industries. The share of business service in aggregate exports of Korea is significantly smaller than those of many developed countries, such as United States, United Kingdom, France, etc., and the industry is still strongly domestic demand-oriented.

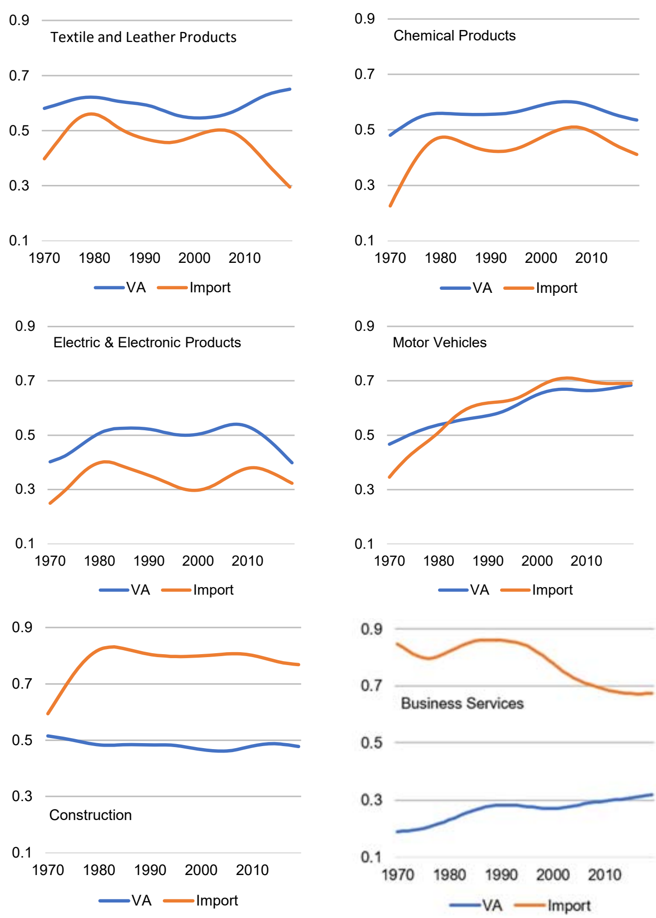
Figure 6. Direct and Indirect Factor Income Shares: Six Selected Industries



Note: D=direct, I=indirect, VA=value-added, M=imports.

The shares of indirect income in total income of the same group of industries for 1970-2019 are given in Figure 7. Note that in textile and leather products, chemical products, and electric and electronic products industries, the I/(D+I) for domestic VA is higher than that for foreign VA, while it is the opposite in the service industries. However, it is not easy to make a clear evaluation about the relative strength of domestic outsourcing and global offshoring because the share of direct imports is too small in the service sector. Again, we observe an M shape, clear or vague, in Figure 7.

Figure 7. Share of Indirect Factor Income ((I/D+I) Ratio): Six Selected Industries



Note: VA=value-added, M=imports.

4 Conclusion

In this paper, we derived a method to decompose gross output or total input in an alternative way at aggregate and sectoral levels. Total input consists of the expenditures on domestic IIs, imported IIs and primary production factors, i.e., VA. The method of the paper is motivated from the fact that domestic IIs are produced by domestic firms, and thus, again, can be decomposed into the same three components following the input structures of the economy or the industries. The same decomposition can be repeatedly applied, and finally we can decompose the expenditure on domestic IIs into the expenditures on VA and imported IIs. These two can be named 'indirect' VA and imports in the sense that they are not the initial, or 'direct,' expenditures on VA and imports, but VA and imports generated by the demand for domestic IIs.

Therefore, gross output can be decomposed into direct and indirect VA and imports. Considering that imports are exports for the trade partner countries and that exports are a part of the final demand, we can regard the imports as the VA in the foreign sector. In conclusion, the paper derived a method to decompose total input or gross output into direct and indirect VA and imports, or equivalently, domestic VA and foreign VA.

The method of the paper can be highly advantageous and useful because the decomposition results can be readily compared among countries, among years, among industries, and among groups of industries. We need IO tables organized in the same industry classification and the same layout for the purpose. The method requires an elementary IO analysis used for computing the impact of final demand shocks on supply-side variables such as gross output, VA, imports and employment. The main concept of the paper's method is to apply the IO analysis to domestic IIs.

In this paper, we used 22 IO tables of Korea for 1970-2019 rearranged according to the same 26-industry classification and the same layout with four VA components. We applied the method to these tables and obtained the decompositions at aggregate level, at sector level, and at industry level. The decomposition results are believed to reflect the characteristics of the industries, the environment of both domestic and global markets, and their changes during the time. The method of the paper can be useful for studying individual industries and for policy studies. While there are a few previous studies to estimate factor income shares at the sectoral level, it is conjectured that this paper is the first to decompose entire gross output into factor incomes.

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Appendix. Proofs of (7)~(9)

Proof of (7)

$$\begin{aligned}
\boldsymbol{o}'\boldsymbol{z}_j^d &= \sum_{i=1}^{n} z_{ij}^d \left(\sum_{l=1}^{n} a_{li}^d + \sum_{l=1}^{n} a_{li}^m + \sum_{k=1}^{p} a_{ki}^v \right) \\
&= \sum_{i=1}^{n} z_{ij}^d \sum_{l=1}^{n} a_{li}^d + \sum_{i=1}^{n} z_{ij}^d \sum_{l=1}^{n} a_{li}^m + \sum_{i=1}^{n} z_{ij}^d \sum_{k=1}^{p} a_{ki}^v \\
&= \sum_{l=1}^{n}\sum_{i=1}^{n} a_{li}^d z_{ij}^d + \sum_{l=1}^{n}\sum_{i=1}^{n} a_{li}^m z_{ij}^d + \sum_{k=1}^{p}\sum_{i=1}^{n} a_{ki}^v z_{ij}^d \\
&= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}' \begin{bmatrix} a_{11}^d & a_{12}^d & \cdots & a_{1n}^d \\ a_{21}^d & a_{22}^d & \cdots & a_{2n}^d \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^d & a_{n2}^d & \cdots & a_{nn}^d \end{bmatrix} \begin{bmatrix} z_{1j}^d \\ z_{2j}^d \\ \vdots \\ z_{nj}^d \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}' \begin{bmatrix} a_{11}^m & a_{12}^m & \cdots & a_{1n}^m \\ a_{21}^m & a_{22}^m & \cdots & a_{2n}^m \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^m & a_{n2}^m & \cdots & a_{nn}^m \end{bmatrix} \begin{bmatrix} z_{1j}^d \\ z_{2j}^d \\ \vdots \\ z_{nj}^d \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}' \begin{bmatrix} a_{11}^v & a_{12}^v & \cdots & a_{1n}^v \\ \vdots & \vdots & & \vdots \\ a_{p1}^v & a_{p2}^v & \cdots & a_{pn}^v \end{bmatrix} \begin{bmatrix} z_{1j}^d \\ z_{2j}^d \\ \vdots \\ z_{nj}^d \end{bmatrix} \\
&= \mathbf{o}'\mathbf{A}^d\mathbf{z}_j^d + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d + \mathbf{o}_p'\mathbf{A}^v\mathbf{z}_j^d \ .
\end{aligned}$$

In fact, this result could have been easily obtained by substituting (3) into $\mathbf{o}'\mathbf{z}_j^d$.

$$\mathbf{o}'\mathbf{z}_j^d = (\mathbf{o}'\mathbf{A}^d + \mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\mathbf{A}^v)\mathbf{z}_j^d$$

by (3) $\mathbf{o}'\mathbf{A}^d + \mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\mathbf{A}^v = \mathbf{o}'$

$$= \mathbf{o}'\mathbf{A}^d\mathbf{z}_j^d + \mathbf{o}'\mathbf{A}^m\mathbf{z}_j^d + \mathbf{o}_p'\mathbf{A}^v\mathbf{z}_j^d \ .$$

Proof of (8)

The right-hand side of (8)

$$= \mathbf{o}'\mathbf{Z}^m + \mathbf{o}'\mathbf{A}^m(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{Z}^d + \mathbf{o}_p'\ \mathbf{V} + \mathbf{o}_p'\mathbf{A}^v\ (\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{Z}^d$$

$$= (\mathbf{o}'\mathbf{Z}^m + \mathbf{o}_p'\ \mathbf{V}) + (\mathbf{o}'\mathbf{A}^m(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{Z}^d + \mathbf{o}_p'\mathbf{A}^v\ (\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{Z}^d)$$

$$= (\mathbf{x}' - \mathbf{o}'\mathbf{Z}^d) + (\mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\mathbf{A}^v\ )(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{Z}^d$$

by (1) $\mathbf{o}'\mathbf{Z}^d + \mathbf{o}'\mathbf{Z}^m + \mathbf{o}_p{}'\mathbf{V} = \mathbf{x}'$

$$= (\mathbf{x}' - \mathbf{o}'\mathbf{Z}^d) + (\mathbf{o}' - \mathbf{o}'\mathbf{A}^d)(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{Z}^d$$

by (3) $\mathbf{o}'\mathbf{A}^d + \mathbf{o}'\mathbf{A}^m + \mathbf{o}_p{}'\mathbf{A}^v = \mathbf{o}'$

$$= (\mathbf{x}' - \mathbf{o}'\mathbf{Z}^d) + \mathbf{o}'(\mathbf{I}-\mathbf{A}^d)(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{Z}^d$$

$$= \mathbf{x}' - \mathbf{o}'\mathbf{Z}^d + \mathbf{o}'\mathbf{Z}^d$$

$$= \mathbf{x}'.$$

Proof of (9)

The right-hand side of (9)

$$= \mathbf{o}'\mathbf{A}^m + \mathbf{o}'\mathbf{A}^m(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{A}^d + \mathbf{o}_p'\ \mathbf{A}^v + \mathbf{o}_p'\mathbf{A}^v\ (\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{A}^d$$

$$= (\mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\ \mathbf{A}^v) + (\mathbf{o}'\mathbf{A}^m(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{A}^d + \mathbf{o}_p'\mathbf{A}^v\ (\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{A}^d)$$

$$= (\mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\ \mathbf{A}^v) + (\mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\mathbf{A}^v\ )(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{A}^d$$

$$= (\mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\ \mathbf{A}^v) + (\mathbf{o}' - \mathbf{o}'\mathbf{A}^d)(\mathbf{I}-\mathbf{A}^d)^{-1}\mathbf{A}^d$$

by (3) $\mathbf{o}'\mathbf{A}^d + \mathbf{o}'\mathbf{A}^m + \mathbf{o}_p{}'\mathbf{A}^v = \mathbf{o}'$

$$= (\mathbf{o}'\mathbf{A}^m + \mathbf{o}_p'\ \mathbf{A}^v) + \mathbf{o}'\mathbf{A}^d$$

$$= \mathbf{o}'.$$

by (3) $\mathbf{o}'\mathbf{A}^d + \mathbf{o}'\mathbf{A}^m + \mathbf{o}_p{}'\mathbf{A}^v = \mathbf{o}'$.